PHY 484 Week 8: The Dirac Spin-$\frac{1}{2}$ Formalism

Informal derivation of the Klein-Gordon equation:

Quantizing the Schrödinger equation to relativity:

$$E = \frac{p^2}{2m} + V \xrightarrow[\substack{p \to -i\hbar\nabla \\ E \to i\hbar\frac{\partial}{\partial t}}]{} \left[-\frac{\hbar^2}{2m}\nabla^2 + V\right]\psi = i\hbar\frac{\partial\psi}{\partial t}.$$

Schrödinger

Similarly, $E^2 = m^2c^4 + p^2c^2$ $\quad (p^\mu p_\mu - m^2c^2 = 0)$

$$\xrightarrow[p_\mu \to i\hbar\partial_\mu]{} \left[-\hbar^2\partial^\mu\partial_\mu - m^2c^2\right]\psi = 0$$

Klein Gordon
Spin-0

KG equation describes spin-0 particles. Or:

$$(\Box^2 + m^2)\psi = 0 \qquad (c = \hbar = 1)$$

Fermions: factoring the Klein-Gordon equation

Case 0 : $p^\mu = (p^0, \vec{0})$. Easy, since $(p^{0 2} - m^2) = 0$

implies $(p^0 + m)(p^0 - m) = 0$.

which is satisfied by $p^0 = \pm m$ (c).

Case 1 : $p^\mu = (p^0, \vec{p})$. Less straightforward.

$$(p^\mu p_\mu - m^2c^2) = (\beta^\kappa p_\kappa + mc)(\gamma^\lambda p_\lambda - mc)$$

$$= \beta^\kappa \gamma^\lambda p_\kappa p_\lambda - m^2c^2 - mc(\beta^\kappa p_\kappa - \gamma^\lambda p_\lambda)$$

So we require $\gamma^\kappa = \beta^\kappa$, no terms linear in p^μ.

Hence $p^\mu p_\mu = \gamma^\kappa \gamma^\lambda p_\kappa p_\lambda$, or

$$(p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2 = (\gamma^0)^2 (p_0)^2 + (\gamma^1)^2 (p_1)^2 + (\gamma^2)^2 (p_2)^2 + (\gamma^4)^2 (p_4)^2$$

$$+ (\gamma^0\gamma^1 + \gamma^1\gamma^0) p_0 p_1 + (\gamma^0\gamma^2 + \gamma^2\gamma^0) p_0 p_2$$

$$+ (\gamma^0\gamma^3 + \gamma^3\gamma^0) p_0 p_3 + \ldots$$

which implies $(\gamma^0)^2 = 1$, $(\gamma^{1,2,3})^2 = -1$, $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0$ $(\mu \neq \nu)$.

$$\Longrightarrow \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

In general (see QFT for proper derivation)

$$\gamma^0 \equiv \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

σ_i = pauli matrices

are 4×4 matrices (are not elements of a four-vector).

Thus, with these properties,

$$p^\mu p_\mu - m^2 = (\gamma^\mu p_\mu + m)(\gamma^\mu p_\mu - m) = 0$$

and we can choose either factor to define as the Dirac equation. Under quantization, $p_\mu \to i\hbar \partial_\mu$, which now operates on a four-component Dirac Bispinor $\psi = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$.

That is, $(i\hbar \not{\partial} - m)\psi = 0$ $\quad \not{\partial} \equiv \gamma^\mu \partial_\mu$

ψ is not a four-vector - it transforms differently under boosts and rotations in 4-space.

That is,

$$\psi \to \psi' = S\psi \quad \text{where} \quad S = \begin{pmatrix} a_+ & a_-\sigma_i \\ a_-\sigma_i & a_+ \end{pmatrix}$$

4x4 matrix

and $i = 1,2,3$ represents the boost axis,

$a_\pm = \pm\sqrt{(\gamma \pm 1)/2}$ (γ is relativistic correction)

Under these transformations, the contraction $\psi^\dagger\psi$ is <u>not</u> a Lorentz invariant, but defining the Dirac Adjoint $\bar{\psi} = \psi^\dagger\beta$, then $\bar{\psi}\psi$ is a Lorentz invariant.

Similarly, under a parity transformation, we have $\psi' = \beta\psi$ which immediately shows that $\bar{\psi}\psi$ is invariant under parity since $\beta^2 = 1$.

Define the "fifth gamma matrix" $\gamma^5 = i\beta\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and the quantity $\bar{\psi}\gamma^5\psi$ is now a pseudoscalar — Lorentz invariant but introduces a minus sign under parity

$$(\bar{\psi}\gamma^5\psi)' \to -\bar{\psi}\gamma^5\psi.$$

We also have $\{\gamma^\mu, \gamma^5\} = 0$, $(\gamma^5)^2 = 1$, $\gamma^{5\dagger} = \gamma^5$.

There are a total of 16 bilinear forms:

	Quantity	Components	Parity
Scalar	$\bar{\psi}\psi$	1	+
Pseudoscalar	$\bar{\psi}\gamma^5\psi$	1	−
Vector	$\bar{\psi}\gamma^\mu\psi$	4	Spatial −
Pseudovector	$\bar{\psi}\gamma^5\gamma^\mu\psi$	4	Spatial +
Tensor	$\bar{\psi}\sigma^{\mu\nu}\psi$	6	

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$.

Stationary Spinors of the Dirac Equation

That is, $(i\hbar\not{\partial} - mc)\psi = 0$ such that $\nabla\psi = (0,0,0)$

($\vec{p} = 0$, since $p^\mu \to i\hbar\partial^\mu$ under quantization)

Then $\frac{i\hbar}{c}\beta\frac{\partial\psi}{\partial t} - mc\psi = 0$.

In matrix form,

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \partial\psi_+/\partial t \\ \partial\psi_-/\partial t \end{pmatrix} = -i\frac{mc^2}{\hbar}\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

where each $\psi_\pm$ are 2-component spinors.

Expanding and solving the ODE yields

$$\boxed{\begin{aligned} \psi_+(t) &= e^{-i(mc^2/\hbar)t}\,\psi_+(0) \\ \psi_-(t) &= e^{i(mc^2/\hbar)t}\,\psi_-(0) \end{aligned}}$$

forward in time (particle)

backwards in time (antiparticle)

So ψ_+ describes electrons $\left(\psi_+(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$ with spin up/down

and ψ_- describes positrons $\left(\psi_-(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$ with spin up/down

Non-Stationary States of Dirac Equation:

or $e^{-i\vec{k}\cdot\vec{r}}$ in normal person notation

We seek plane wave solutions of the form $\Psi(\vec{r},t) = a\, e^{-\frac{i}{\hbar}(Et - \vec{p}\cdot\vec{r})} u(E,\vec{p})$,

So the Dirac equation becomes

$$(\not{p} - mc)u = 0$$

when substituted in. In matrix form, this is equivalently

$$\begin{pmatrix} \frac{E}{c} - mc & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -\frac{E}{c} - mc \end{pmatrix} \begin{pmatrix} u_+ \\ u_- \end{pmatrix} = 0.$$

This gives two coupled equations in u_+ and u_-:

$$u_+ = \frac{c}{E - mc^2} (\vec{p}\cdot\vec{\sigma}) u_-$$

$$u_- = \frac{c}{E + mc^2} (\vec{p}\cdot\vec{\sigma}) u_+$$

which implies $\boxed{u_\pm = \frac{c^2}{E^2 - m^2c^4} (\vec{p}\cdot\vec{\sigma})^2 u_\pm .}$

The factor $\vec{p}\cdot\vec{\sigma}$ is the matrix polynomial

$$\vec{p}\cdot\vec{\sigma} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

and since $\sigma^2 = 1$, $(\vec{p}\cdot\vec{\sigma})^2 = |\vec{p}|^2$ (easy to check).

We then find that the solution $u_I = \frac{c^2}{E^2 - m^2c^4} |\vec{p}|^2 u_I$

requires that $E^2 = m^2c^4 + p^2c^2$ (both $+,-$ solutions!)

This yields the explicit solutions

$$u_1 = N \begin{pmatrix} 1 \\ 0 \\ \frac{cp_3}{E+mc^2} \\ \frac{c(p_1+ip_2)}{E+mc^2} \end{pmatrix} \qquad u_2 = N \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_1+ip_2)}{E+mc^2} \\ \frac{cp_3}{E+mc^2} \end{pmatrix}$$

(particles) $E>0$

$$v_1 = N \begin{pmatrix} \frac{c(p_1-ip_2)}{E+mc^2} \\ \frac{-cp_3}{E+mc^2} \\ 1 \\ 0 \end{pmatrix} \qquad v_2 = N \begin{pmatrix} \frac{cp_3}{E+mc^2} \\ \frac{c(p_1+ip_2)}{E+mc^2} \\ 0 \\ 1 \end{pmatrix}$$

(antiparticles) $E<0$

The relativistic wavefunction normalization requires that $u_I^{\dagger} u_I = \frac{2|E|}{c}$

which, under expansion of $u_{1,2} / v_{1,2}$, has $N = \sqrt{(|E| + mc^2)/c}$

Spin Operators for Bispinors:

$$\vec{S} = \frac{\hbar}{2}\vec{\Sigma} \quad ; \quad \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

When S_z axis choice aligns with that of the momentum direction $\vec{p}$, then

u_1, u_3 are spin-up ; u_2, u_4 are spin-down.

To represent antifermion solutions, we flip the signs of both E and $\vec{p}$ in

v_1 and v_2 (already specified).

u solves $(\not{p} - mc)u = 0$

v solves $(\not{p} + mc)v = 0$.